# A New Finite Element for Static and Dynamic Analysis of Cracked Composite Beams

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FRACTURE & DYNAMICS
PAPER NO. 43

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# A NEW FINITE ELEMENT FOR STATIC AND DYNAMIC ANALYSIS OF CRACKED COMPOSITE BEAMS

(Submitted to An International Journal Computers & Structures)

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**Abstract** - A new beam finite element with a single nonpropagating one-edge open crack located in its mid-length is formulated for static and dynamic analysis of cracked composite beam-like structures. The element includes two degrees of freedoms at each of the three nodes: a transverse deflection and an independent rotation respectively. The exemplary numerical calculations illustrating variations of static deformations and a fundamental bending natural frequency of composite cantilever beam caused by a single crack are presented. The element proves to be accurate and versatile. The compatibility with plate and shell elements as a stiffener is assured through the use of simple nodal variables of $C^0$-type. The presented method of creating the element makes it possible to construct beam finite elements with other types of cracks (double-edge, internal etc.) provided that stress intensity factors for a given type of crack are known.

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## 1. INTRODUCTION

High speed machinery and lightweight structures require high strength-to-weight ratios. For this reason, in recent years, the use of anisotropic reinforced composites, where strength-to-weight ratios is very high, has increased substantially in the fields of mechanical and civil engineering - see for example the textbook of Vinson and Chou [1].

Cracks occurring in structural elements are responsible for local stiffness variations [2], which in consequence affect their dynamic characteristics. This problem has been a subject of many papers, the review of which is given by Wauer [3], but only several papers has been devoted the changes of the dynamic characteristics of composite constructional elements. Adams *et al.* [4], found that damage in specimens fabricated from fiber reinforced plastics could be detected by reduction in natural frequencies and an increase in damping. Cawley and Adams [5], successfully tested the frequency measurement principle on composite structures made in the presence of damage. Nikpour and Dimarogonas [6], presented the local compliance matrix for unidirectional composite materials. They have shown that the interlocking deflection modes are enhanced as a function of the degree of anisotropy in composites. The effect of cracks upon buckling of an edge-notched column for isotropic and anisotropic composites has been studied by Nikpour [7]. He indicated that the instability increases with the column slenderness and the crack length. In addition he has shown that the material anisotropy conspicuously reduces the load-carrying capacity of an externally cracked member. Recently, Manivasagam and Chandrasekaran [8], have presented results of experimental investigations upon the reduction effect of the fundamental frequency of layered composite materials with damage in the form of cracks.

In the presented paper there has been made an attempt to work out a composite beam

finite element with nonpropagating one-edge open crack. It has been assumed that the crack changes only stiffness of the element whereas the mass of the element is unchanged. The element has been tested by numerical calculations, the results of which has been compared with results other investigators. The influence of the crack depth, the volume fraction of fiber and also the angle of fiber upon changes the static deflection and the fundamental natural frequency of the composite cantilever beam has been studied.

## 2. GENERAL DESCRIPTION OF THE NONCRACKED ELEMENT

The geometry of the presented element is shown in Fig.1. The element has three nodes at x=-0.5L, 0, 0.5L. At each node there are two degrees of freedom, which are the transverse displacements $q_1$, $q_3$, $q_5$ and independent rotations $q_2$, $q_4$, $q_6$.

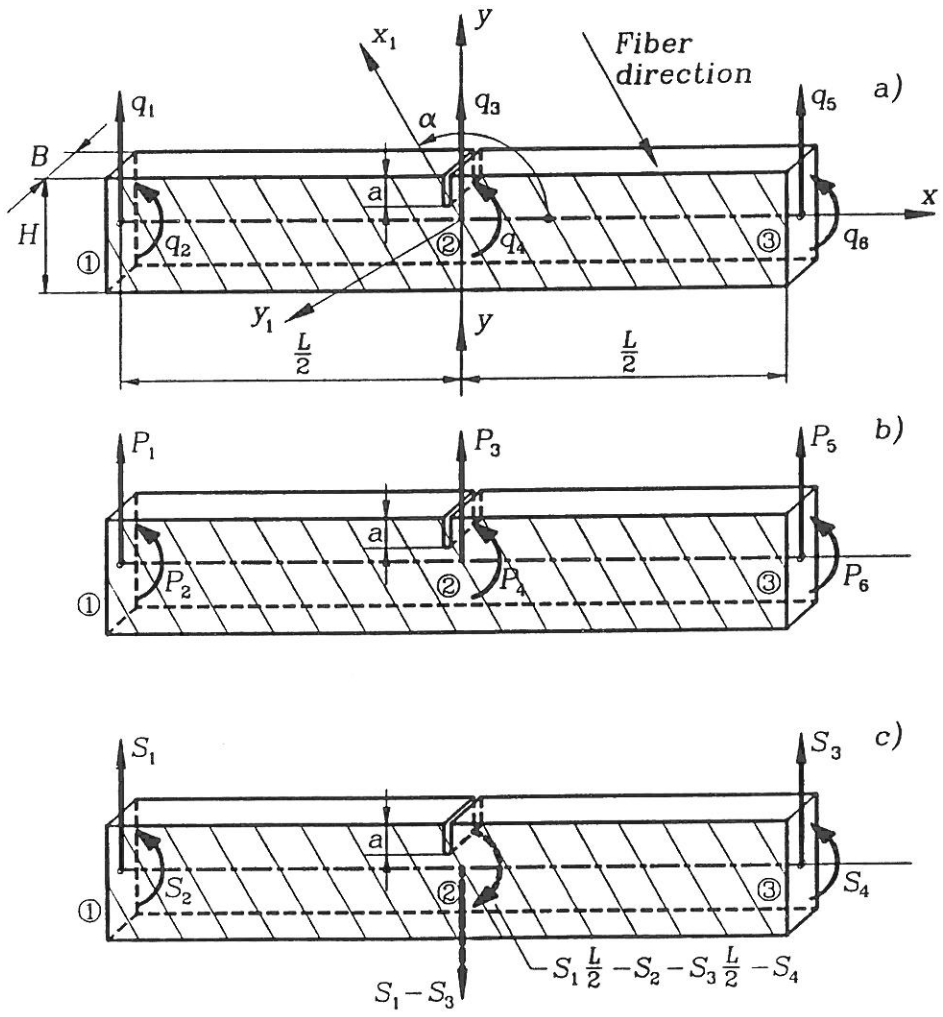


Fig.1

Neglecting warping the displacements $u_x$, $u_y$ of a point of the element can be expressed as [9]

$$u_x(x,y) = -y\phi(x), \tag{1.a}$$

$$u_y(x,y) = v(x), \tag{1.b}$$

where $\phi(x)$ is the rotation and $v(x)$ denotes the transverse displacement.

Upon differentiation, the strains in the element are obtained as

$$\varepsilon_x = -y \frac{\partial \phi(x)}{\partial x}, \tag{2.a}$$

$$\gamma_{xy} = \frac{\partial v(x)}{\partial x} - \phi(x) . \tag{2.b}$$

In the finite element method, the transverse displacement $v(x)$ is assumed as cubic polynomials in x, while the independent rotation $\phi(x)$ can be expressed by quadratic polynomials. Hence, considering only the bending in the x-y plane the variables $v(x)$ and $\phi(x)$ are given by the following relations

$$v(x) = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3, \tag{3.a}$$

$$\phi(x) = \beta_5 + \beta_6 x + \beta_7 x^2. \tag{3.b}$$

Assuming the shear strain variation to be linear as it proposed by Tessler and Dong [10], one receives relation between constant $\beta_4$ and $\beta_7$ in the form

$$\beta_7 = 3\beta_4, \tag{4}$$

hence, the transverse displacement $v(x)$ and the independent rotation $\phi(x)$ can be written as

$$v(x) = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3, \tag{5.a}$$

$$\phi(x) = \beta_5 + \beta_6 x + 3\beta_4 x^2. \tag{5.b}$$

Take into account the boundary conditions at nodes of the element, the variables $v(x)$ and $\phi(x)$ can be expressed in terms of the element degrees of freedom

$$v(x)=q_3 + \left(-\frac{q_1}{L} - \frac{q_2}{6} + \frac{q_4}{3} + \frac{q_5}{L} - \frac{q_6}{6}\right)x + \left(\frac{2q_1}{L^2} - \frac{4q_3}{L^2} + \frac{2q_5}{L^2}\right)x^2 + \left(\frac{2q_2}{3L^2} - \frac{4q_4}{3L^2} + \frac{2q_6}{3L^2}\right)x^3, \tag{6.a}$$

$$\phi(x) = q_4 + \left(-\frac{q_2}{L} + \frac{q_6}{L}\right)x + \left(\frac{2q_2}{L^2} - \frac{4q_4}{L^2} + \frac{2q_6}{L^2}\right)x^2 , \tag{6.b}$$

Substitution relations (6.a-b) into (1.a-b) yields the displacements $u_x$, $u_y$ of a point of the element in the form

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \mathbb{N} \begin{pmatrix} q_1 \\ \vdots \\ q_6 \end{pmatrix} , \tag{7}$$

where $\mathbb{N}$ denotes the shape function matrix of the element in the form

$$\mathbb{N} = \left[\begin{array}{c|c|c|c|c|c} 0 & (\frac{x}{L} - \frac{2x^2}{L^2})y & 0 & (\frac{4x^2}{L^2} - 1)y & 0 & -(\frac{2x^2}{L^2} + \frac{x}{L})y \\ \hline \frac{2x^2}{L^2} - \frac{x}{L} & \frac{2x^3}{3L^2} - \frac{x}{6} & 1 - \frac{4x^2}{L^2} & \frac{x}{3} - \frac{4x^3}{3L^2} & \frac{x}{L} + \frac{2x^2}{L^2} & \frac{2x^3}{3L^2} - \frac{x}{6} \end{array}\right], \tag{8}$$

In the similar fashion, substitution (6.a-b) into (2.a-b) yields the strains in terms of the element degrees of freedom as

$$\begin{pmatrix} \varepsilon_x \\ \gamma_{xy} \end{pmatrix} = \mathbb{B} \begin{pmatrix} q_1 \\ \vdots \\ q_6 \end{pmatrix} , \tag{9}$$

where $\mathbb{B}$ denotes the strains-nodal displacements relation matrix in the form

$$\mathbb{B} = \left[\begin{array}{c|c|c|c|c|c} 0 & \frac{y}{L} - \frac{4xy}{L^2} & 0 & \frac{8xy}{L^2} & 0 & -\frac{4xy}{L^2} - \frac{y}{L} \\ \hline \frac{4x}{L^2} - \frac{1}{L} & \frac{x}{L} - \frac{1}{6} & -\frac{8x}{L^2} & -\frac{2}{3} & \frac{1}{L} + \frac{4x}{L^2} & -\frac{x}{L} - \frac{1}{6} \end{array}\right], \tag{10}$$

## 3. INERTIA MATRIX OF THE CRACKED ELEMENT

Because, it has been assumed that the crack occurring in the element not change the mass of its, the inertia matrix has the same form like in the case of the noncracked one.

The inertia matrix of the noncracked element $\mathbb{M}_e$ is calculating from the commonly known equation, Zienkiewicz [11]

$$\mathbb{M}_e = \rho \int_v \mathbb{N}^T \mathbb{N} \, dv, \tag{11}$$

where $\rho$ is the mass density and v is the volume of the element.

Substitution (8) into (11) yields the inertia matrix of the presented element in the closed form

$$\mathbb{M}_e = \frac{\rho BHL}{3780} \left[\begin{array}{c|c|c|c|c|c} 504 & & & & & \\ \hline 21L & 42H^2+2L^2 & & & \text{sym.} & \\ \hline 252 & 0 & 2016 & & & \\ \hline -42L & 21H^2-4L^2 & 0 & 168H^2+8L^2 & & \\ \hline -126 & -21L & 252 & 42L & 504 & \\ \hline 21L & -10.5H^2+2L^2 & 0 & 21H^2-4L^2 & -21L & 42H^2+2L^2 \end{array}\right] \tag{12}$$

where B,H,L are dimensions of the element shown in Fig.1.

## 4. STIFFNESS MATRIX OF THE CRACKED ELEMENT

The stiffness matrix $\mathbb{K}_e$ of the finite element can be calculated by means of the relation, Przemieniecki [12]

$$\mathbb{K}_e = \mathbb{T}^t \, \mathbb{C}^{-1} \, \mathbb{T}, \tag{13}$$

where $\mathbb{T}$ is the transformation matrix of a system of dependent nodal forces $P_i$ (i=1,6) into the system of independent nodal forces $S_i$ (i=1,4) - see Fig.1, $\mathbb{C}^{-1}$ is the inverse of flexibility matrix of the element, t denotes transpose of the matrix.

In the case of the cracked element the flexibility matrix $\mathbb{C}$ is represented by a sum of the flexibility matrix of the noncracked element $\mathbb{C}^o$ and the additional flexibility matrix $\mathbb{C}^1$ caused by the crack

$$\mathbb{C} = \mathbb{C}^o + \mathbb{C}^1, \tag{14}$$

## 5. MATRIX OF TRANSFORMATION

The matrix of transformation $\mathbb{T}$ is calculated using the equation of overall equilibrium for element forces $P_i$ (i=1,6) and $S_i$ (i=1,4) - see Fig.1. The finally form of this matrix can be presented in the following form

$$\mathbb{T}^t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ L/2 & -1 & -L/2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{15}$$

## 6. FLEXIBILITY MATRIX OF THE NONCRACKED ELEMENT

The terms of the flexibility matrix of the noncracked element $\mathbb{C}^o$ can be determined by inversion of the force-displacement equation, Przemieniecki [12]. For the presented element the force-displacement equation has the form

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}, \tag{16}$$

According with Fig.1 the second node of the element is constrained i.e.

$$q_3 = q_4 = 0, \tag{17}$$

Applying the condition (17), the equation (16) can be transformed to the inversion form

$$\begin{bmatrix} q_1 \\ q_2 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{25} & k_{26} \\ k_{51} & k_{52} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{65} & k_{66} \end{bmatrix}^{-1} \begin{bmatrix} P_1 \\ P_2 \\ P_5 \\ P_6 \end{bmatrix} \tag{18}$$

Finally, the flexibility matrix of the noncracked element under a selected independent system is

$$\mathbb{C}^o = \begin{bmatrix} k_{11} & k_{12} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{25} & k_{26} \\ k_{51} & k_{52} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{65} & k_{66} \end{bmatrix}^{-1}. \tag{19}$$

The terms $k_{ij}$ are equal to terms of the stiffness matrix of the noncracked element $\mathbb{K}_e$, which is calculated according with the following formula, Zienkiewicz [11]

$$\mathbb{K}_e = \int_V \mathbb{B}^T \, \mathbb{D} \, \mathbb{B} \, dv, \tag{20}$$

where $\mathbb{D}$ denotes the stresses-strains relation matrix - see Appendix C.

Substitution (10) into (20) yields the terms of the stiffness matrix of the noncracked element in the form

$$k_{11} = k_{55} = 7BH\bar{S}_{33}/3L \ ,$$

$$k_{12} = k_{21} = -k_{56} = -k_{65} = BH\bar{S}_{33}/2 \ ,$$

$$k_{13} = k_{31} = k_{35} = k_{53} = -8BH\bar{S}_{33}/3L \ ,$$

$$k_{14} = k_{41} = k_{36} = k_{63} = -k_{23} = -k_{32} = -k_{45} = -k_{54} = 2BH\bar{S}_{33}/3 \ ,$$

$$k_{15} = k_{51} = BH\bar{S}_{33}/3L \ ,$$

$$k_{16} = k_{61} = -k_{25} = -k_{52} = -BH\bar{S}_{33}/6 \ ,$$

$$k_{22} = k_{66} = BH(7H^2\bar{S}_{11}/36L + L\bar{S}_{33}/9) \ ,$$

$$k_{24} = k_{42} = k_{46} = k_{64} = BH(-2H^2\bar{S}_{11}/9L + L\bar{S}_{33}/9) \ ,$$

$$k_{26} = k_{62} = BH(H^2\bar{S}_{11}/36L - L\bar{S}_{33}/18) \ ,$$

$$k_{33} = 16BH\bar{S}_{33}/3L \ ,$$

$$k_{44} = BH(4H^2\bar{S}_{11}/9L + 4L\bar{S}_{33}/9) \ ,$$

$$k_{34} = k_{43} = 0.$$

where the form $\bar{S}_{11}$ and $\bar{S}_{33}$ is given in the Appendix C.

## 7. ADDITIONAL FLEXIBILITY MATRIX OF THE ELEMENT DUE TO THE CRACK

The terms of the additional flexibility matrix $\mathbb{C}^1$ due to the crack are calculated making use of the relation

$$c_{ij}^1 = \frac{\partial^2 U^1}{\partial S_i \partial S_j} , \tag{21}$$

where $U^1$ is the additional elastic strain energy of the element caused by the crack, $S_i$, $S_j$ are independent nodal forces acting on the element.

The additional elastic strain energy $U^1$ due to the presence of the crack in the analyzed element can be expressed as a function of the stress intensity factors [6]

$$U^1 = \int_A \left( D_1 \sum_{i=1}^{i=4} K_{IIi}^2 + D_{12} \sum_{i=1}^{i=4} K_{IIi} \sum_{i=1}^{i=4} K_{IIIi} + D_2 \sum_{i=1}^{i=4} K_{IIIi}^2 \right) dA \tag{22}$$

where A is the area of the crack, $K_{IIi}$, $K_{IIIi}$ are the stress intensity factors corresponding with two models of the crack evaluation, i denotes independent nodal forces acting on the element, and the coefficients $D_1$, $D_{12}$ and $D_2$ are given by the following relations [6]

$$D_1 = - 0.5\bar{b}_{22} \mathrm{Im}\left( \frac{s_1+s_2}{s_1 s_2} \right) , \tag{23.a}$$

$$D_{12} = \bar{b}_{11} \mathrm{Im}\left( s_1 s_2 \right) , \tag{23.b}$$

$$D_2 = 0.5\bar{b}_{11} \mathrm{Im}\left( s_1 + s_2 \right) . \tag{23.c}$$

The method of calculation the terms $s_1$, $s_2$ and $\bar{b}_{ij}$ is shown in the Appendix B. The variations of coefficients $D_1$, $D_{12}$ and $D_2$ versus the fiber volume fraction and the crack angle are presented in Figs.2-4.

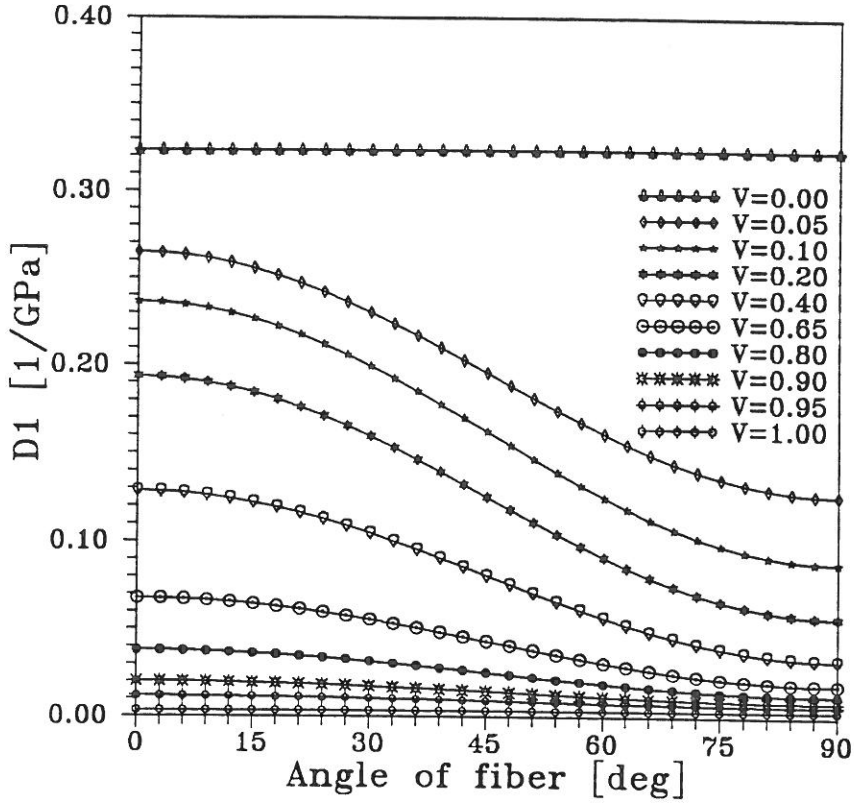
0.40
0.30
0.20
0.10
0.00
D1 [1/GPa]
V=0.00
V=0.05
V=0.10
V=0.20
V=0.40
V=0.65
V=0.80
V=0.90
V=0.95
V=1.00
0
15
30
45
60
75
90
Angle of fiber [deg]

Fig.2

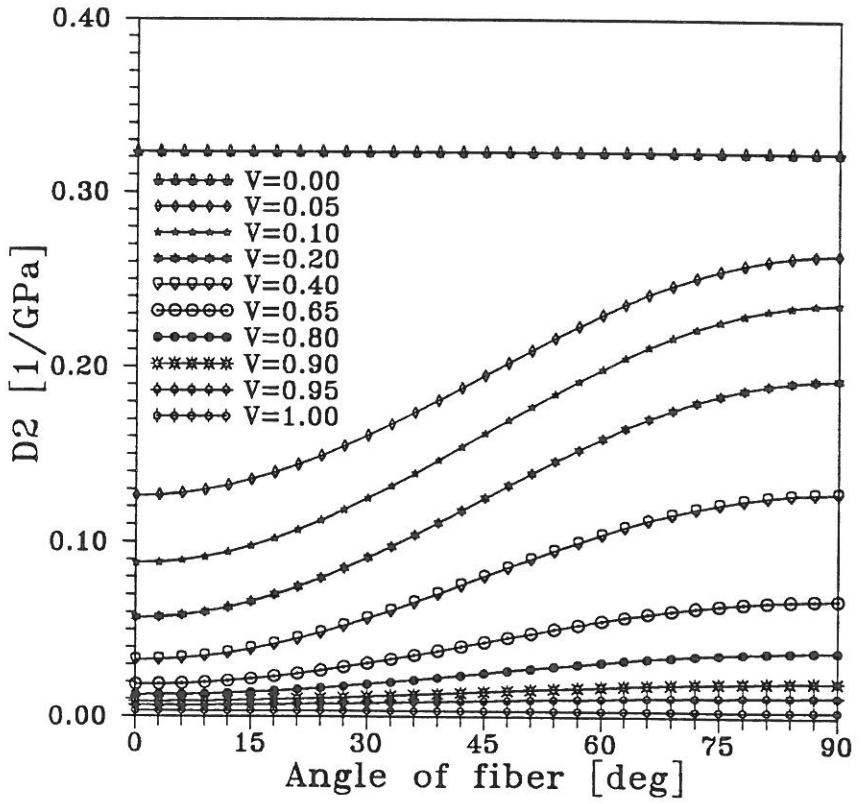
0.40
0.30
0.20
0.10
0.00
D2 [1/GPa]
V=0.00
V=0.05
V=0.10
V=0.20
V=0.40
V=0.65
V=0.80
V=0.90
V=0.95
V=1.00
0
15
30
45
60
75
90
Angle of fiber [deg]

Fig.3

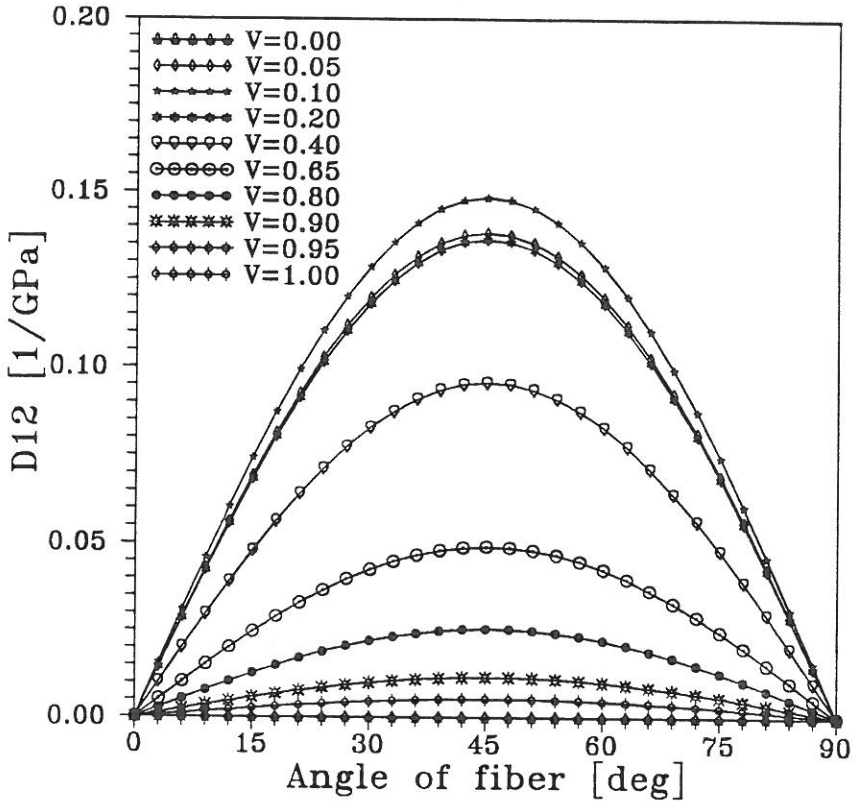


Fig.4

The stress intensity factors can be expressed as a function of independent nodal forces (Krawczuk and Ostachowicz [13], and also Krawczuk [14]). Generally, for anisotropic materials, the stress intensity factors are not equivalent to those of isotropic bodies of the same geometry and loading conditions, expect when the crack tips are sufficiently far away from loading points and the edges of specimen.

For the one-edge crack the nonzero stress intensity factors in function of independent nodal forces are equal

$$K_{I2} = \frac{6S_2}{BH^2} \sqrt{\pi a}\, Y_1 \,, \tag{24.a}$$

$$K_{I4} = \frac{6S_4}{BH^2} \sqrt{\pi a}\, Y_1 \,, \tag{24.b}$$

$$K_{II1} = \frac{S_1}{BH} \sqrt{\pi a}\, Y_2 \tag{24.c}$$

$$K_{II3} = \frac{S_3}{BH} \sqrt{\Pi a}\, Y_2 \ , \tag{24.d}$$

where a is the crack depth, $Y_1$ and $Y_2$ are correction factors.

The correction factors $Y_1$ and $Y_2$ arise from the lack of symmetry and the deformation at the free edges of the beam compared with an infinite plate containing a crack. These factors are nondimensional functions of the relative depth of crack ($\bar{a}$=a/H) and the anisotropic constants of material which may be expressed in terms of the roots of the characteristic equation (Appendix B). In many cases, however, the numerical analysis of highly anisotropic materials demonstrates a very weak correlation between the material anisotropic constants and Y-factors. Denoting these anisotropic perturbations by $C_1(\zeta)$, the Y-factors for isotropic materials given by Tada *et al.* [15], can be expressed as

$$Y_1 = \sqrt{\tan\lambda/\lambda}\,[0.923 + 0.199(1-\sin\lambda)^4]C_1(\zeta)/\cos\lambda \ , \tag{25.a}$$

$$Y_2 = (1.122 - 0.561\bar{a} + 0.085\bar{a}^2 + 0.18\bar{a}^3)/\sqrt{1 - \bar{a}}, \tag{25.b}$$

where $\lambda = \Pi\bar{a}/2$.

The factor $C_1(\zeta)$ for the edge-notched beam can be fitted by a single function [16]

$$C_1(\zeta) = 1.0 + 0.1(\zeta-1) - 0.16(\zeta-1)^2 + 0.002(\zeta-1)^3 \ . \tag{26}$$

where $\zeta = \dfrac{\sqrt{E_{11}E_{22}}}{2G_{12}} - \nu_{12}$, (the material constants $E_{11}$, $E_{22}$, $G_{12}$ and $\nu_{12}$ are described in Appendix A).

Substitution relations (24.a-d), (25.a-b) and (26) into (22) yields the additional flexibility matrix of the element due to the crack as

$$\mathbb{C}^1 = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ & c_{22} & c_{23} & c_{24} \\ & & c_{33} & c_{34} \\ \text{sym.} & & & c_{44} \end{bmatrix} \tag{27}$$

where the following terms of the matrix are equal

$$c_{11} = c_{13} = c_{33} = \frac{2\Pi D_2}{B} \int_0^{\bar{a}} \bar{a}\, Y_2^2\, d\bar{a} ,$$

$$c_{22} = c_{24} = c_{44} = \frac{72\Pi D_1}{BH^2} \int_0^{\bar{a}} \bar{a}\, Y_1^2\, d\bar{a} ,$$

$$c_{12} = c_{14} = c_{23} = c_{34} = \frac{6\Pi D_{12}}{BH} \int_0^{\bar{a}} \bar{a}\, Y_1 Y_2\, d\bar{a} .$$

The changes of integrals as a function of the relative depth of the crack, for graphite-fiber reinforced polyimide from Appendix A (volume fraction of fiber - 10%), are presented in Fig.5.

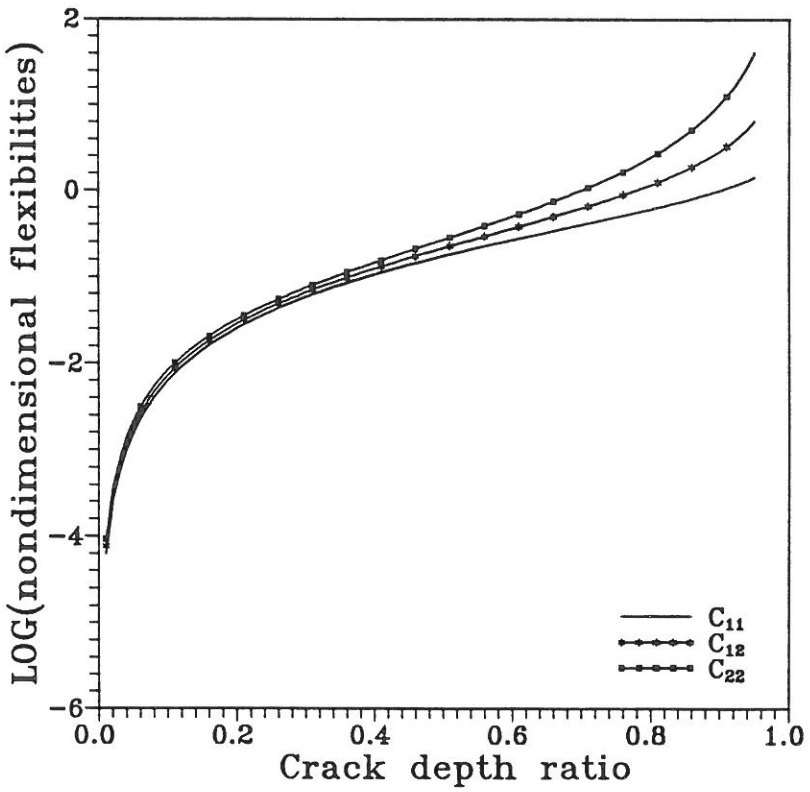


Fig.5

## 8. NUMERICAL STUDIES

The formulation of the element have been evaluated by performing the following examples.

*1. Static deflection of the noncracked composite cantilever beam*

The noncracked composite cantilever beam shown in Fig.6 is subjected to bending force. The material properties of graphite-fiber reinforced polyimide used in the analysis are given in the Appendix A. The calculations were carried out for various values of the angle of fiber and the volume fraction of fiber V.

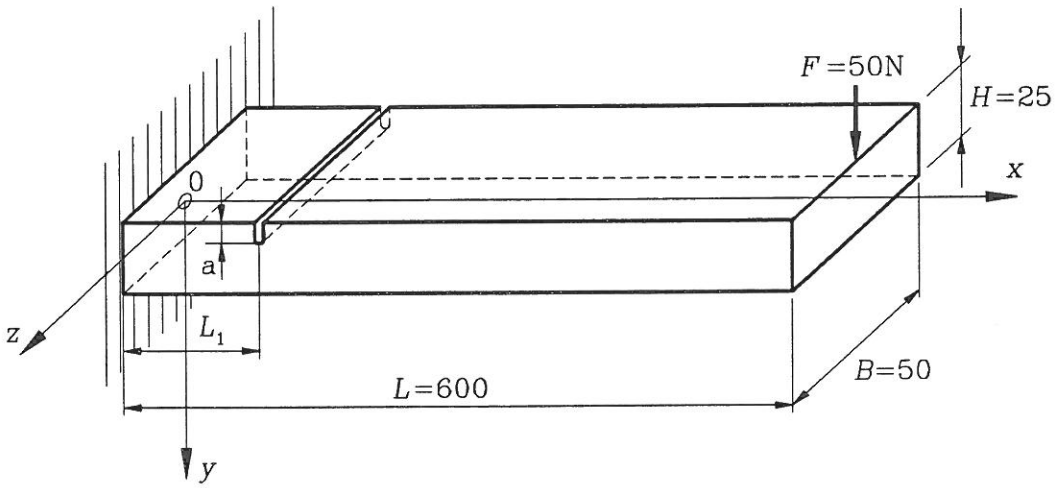


Fig.6

The static deflection of the end of beam obtained by a four element model is compared with analytical solution of Lekhnitskii [17], in Fig.7. It is noted that for all values of the fiber volume fraction V and the fiber angle results are in excellent agreement.

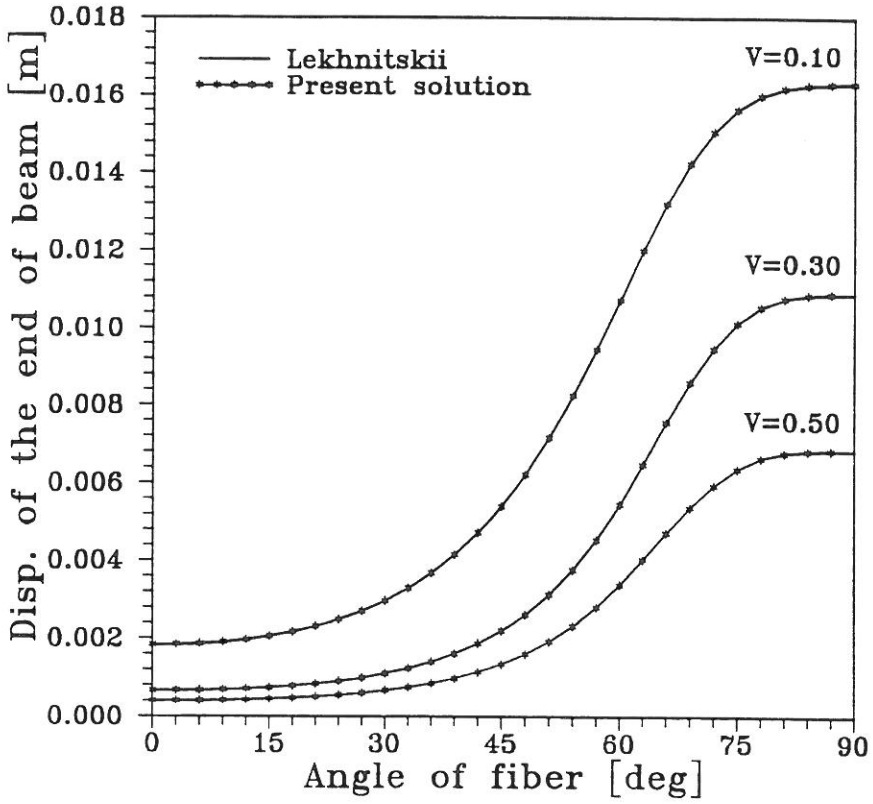


Fig.7

*2. Static deflection of the cracked composite cantilever beam*

The second example was carried out for the cracked composite cantilever beam with the same material properties and dimensions like in the point 1. The nonpropagating one-edge open crack is located 75 mm from the fixed end of the beam. The depth of the crack is various, equal to 0.2, 0.4 and 0.6 of the height of beam, respectively. The model of the beam contains three noncracked beam elements and one element with crack.

Fig.8 shows the relative static deflections of the end of beam as a function of the fiber angle and the relative depth of the crack a/H, for various values of the fiber volume fraction. The relative static deflection is calculating as

$$f_r = \frac{f_c(\alpha)}{f_{nc}(\alpha)} , \tag{28}$$

where $f_c(\alpha)$ denotes the static deflection of the cracked beam as a function of the angle of

fiber, $f_{nc}(\alpha)$ denotes the static deflection of the noncracked beam as a function of the angle of fiber, $\alpha$ is the angle of fiber.

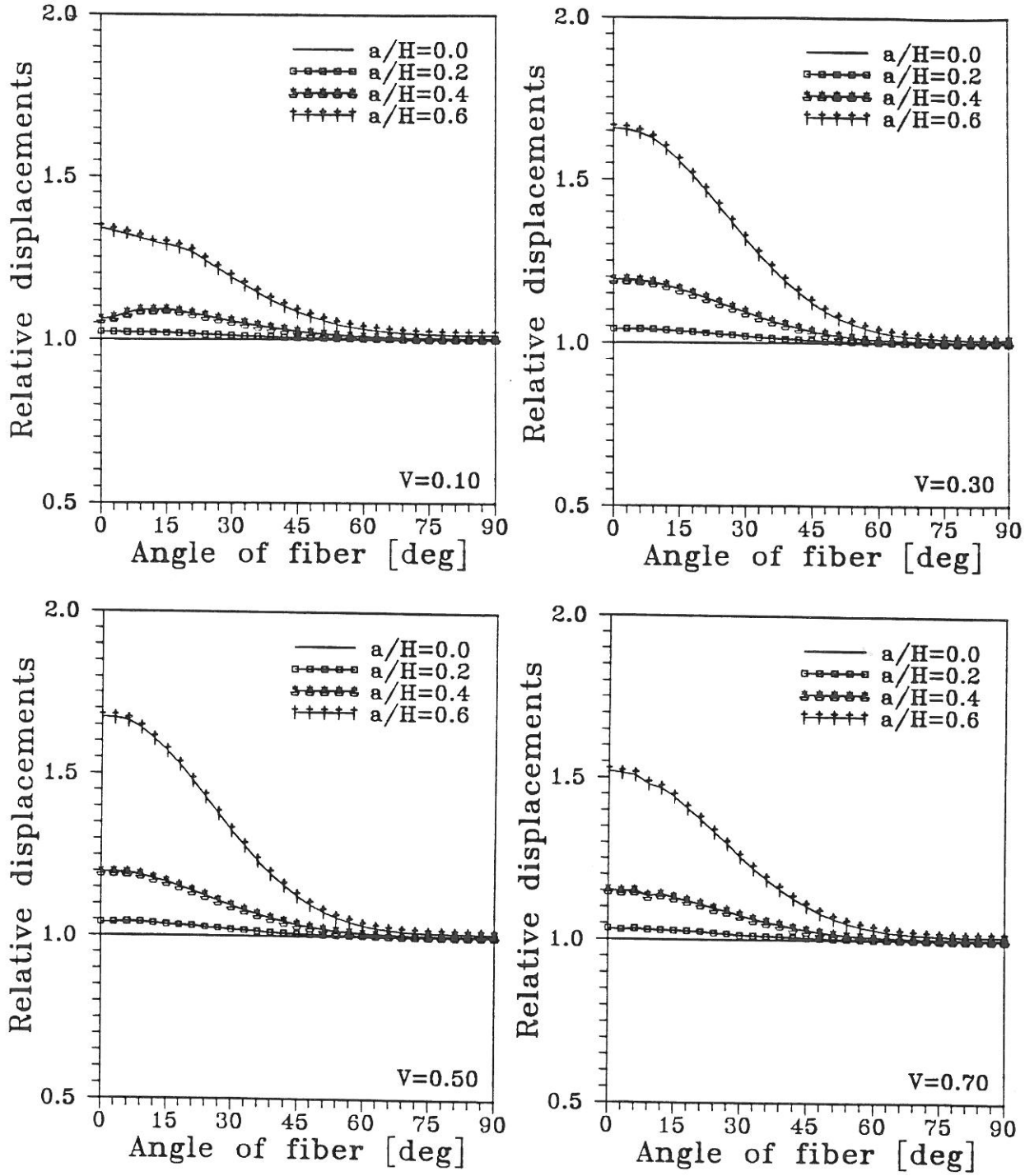


Fig.8

The maximum value of the relative static deflection is observed for the crack perpendiculars to fibers of the composite. When the fiber angle increases the relative static deflection is reduced and practically for the angle greater than 45 deg. the static deflection (even for relatively depth crack a/H=0.4), has the same value like in the case of the noncracked beam.

Fig.9 shows the influence of the volume fraction of fiber on the relative static deflections of the analyzed beam. In this case, the relative static deflection of the beam is calculating as

$$f_r = \frac{f_c(V)}{f_{nc}(V)}, \tag{29}$$

where $f_c(V)$ denotes the static deflection of the cracked beam as a function of the volume fraction of fiber, $f_{nc}(V)$ denotes the static deflection of the noncracked beam as a function of the volume fraction of fiber, V is the volume fraction of fiber.

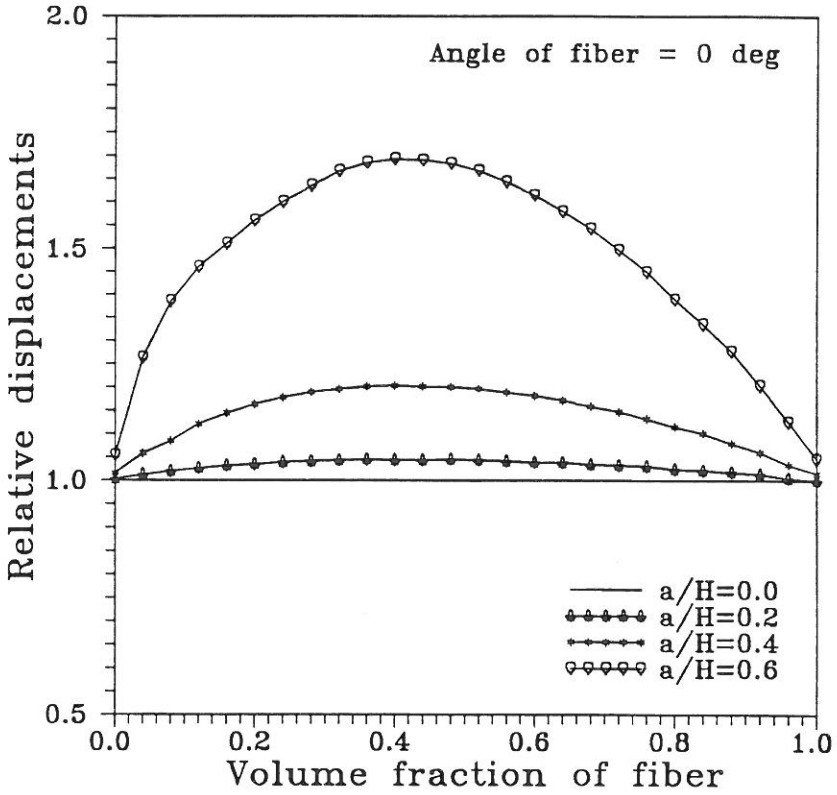


Fig.9

The relative static deflection is strongly dependent on the volume fraction of fiber. The maximum value is achieved at relatively higher fiber fractions (around 40%).

*3. Natural frequencies of the noncracked composite cantilever beam*

In this point the bending natural frequencies of the noncracked composite beam from example 1 were determined. The calculations were carried out for various values of the angle of fiber and the fiber volume fraction V.

The first three nondimensional bending natural frequencies obtained by a four element model are compared with analytical solution given by Vinson and Sierakowski [18], in Fig.10. The frequencies are normalized according with the relation

$$\bar{\omega} = l \sqrt{\omega h \,/\, \sqrt{\bar{S}_{11}/12\rho}} \quad , \tag{30}$$

where l is the length of the beam, h denotes the height of the beam and $\omega$ is the dimensional natural frequency.

It is noted that for first free natural frequencies results are in satisfactory agreement.

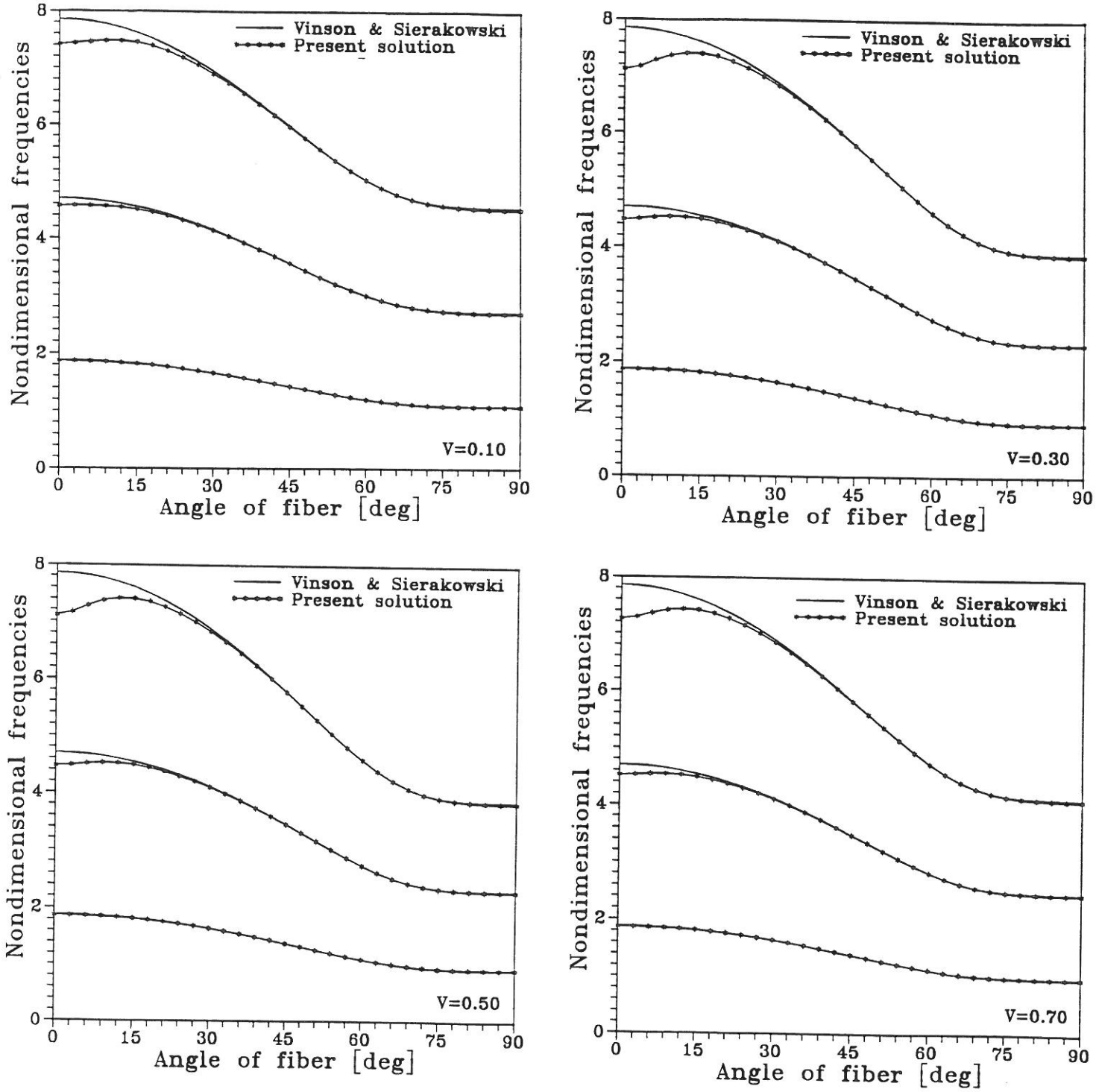


Fig.10

*4. Natural frequencies of the cracked composite cantilever beam*

The last example is devoted to analyze the change of the bending natural frequencies of the cracked composite beam from example 2. The calculations were carried out for various

value of the fiber volume fraction, the fiber angle and the depth of the crack. The model of the beam contains three noncracked beam elements and one element with crack.

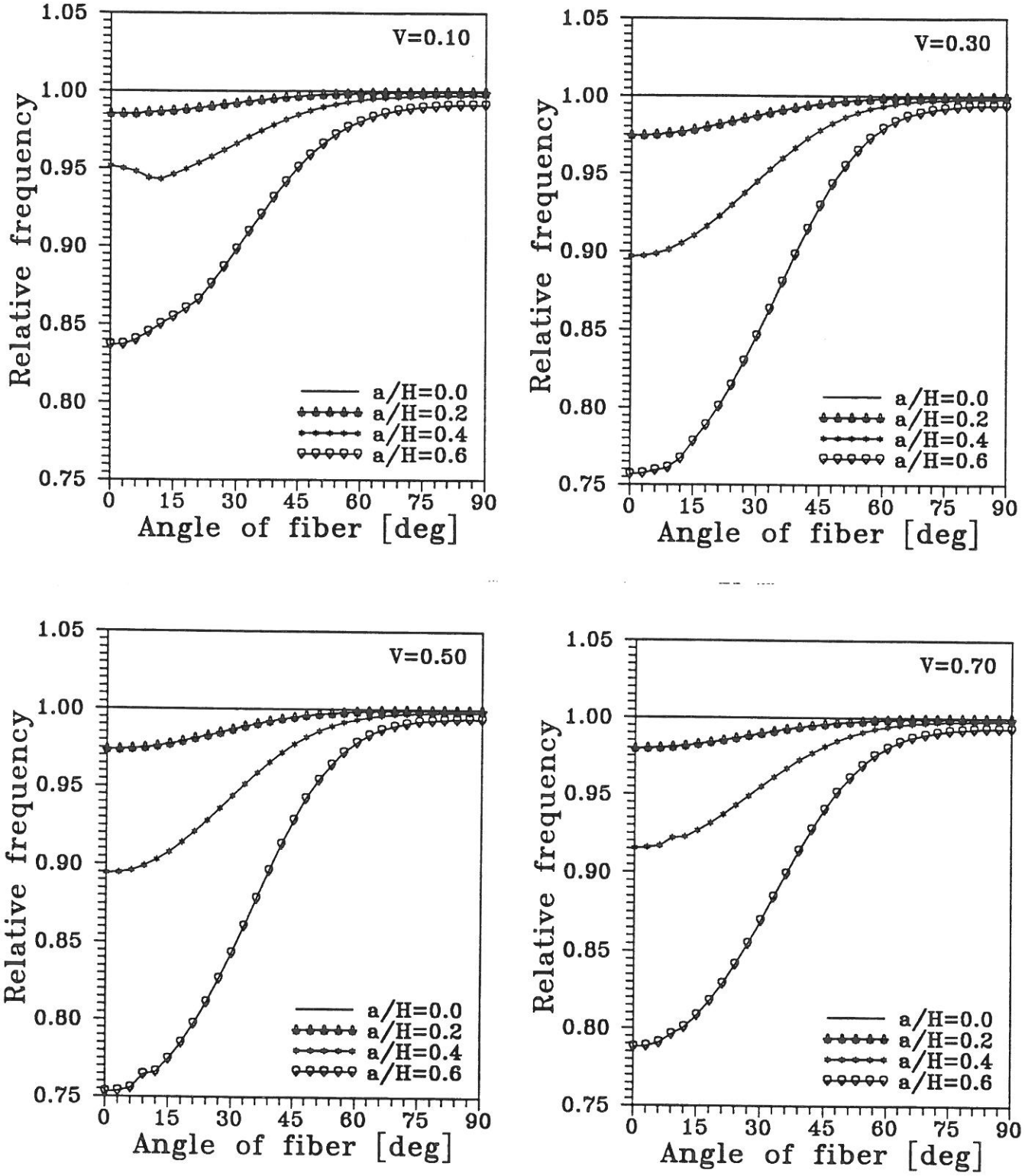


Fig.11

The relative changes of the first bending natural frequency of the beam as a function of the relative depth of crack and the angle of fiber, for several values of the volume fraction of fiber, are shown in Fig.11. The changes are normalized according with the following relation

$$\omega_r = \frac{\omega_c(\alpha)}{\omega_{nc}(\alpha)} , \tag{31}$$

where $\omega_c(\alpha)$ denotes the first bending natural frequency of the cracked beam as a function of the angle of fiber, $\omega_{nc}(\alpha)$ denotes the first bending natural frequency of the noncracked beam as a function of the angle of fiber.

The decrease of the fundamental bending natural frequency is strongest for the crack perpendiculars to the fiber direction. When, the angle of fiber increases this effect decreases and for the angle greater than 45 deg., the first bending natural frequency of the cracked beam has the same value like in the case of the noncracked beam (even for relatively depth crack - a/H=0.4).

Fig.12 shows the influence of the volume fraction of fiber on relative changes of the first bending frequency of the analyzed beam. In this case the relative changes of the natural frequency are calculating as

$$\omega_r = \frac{\omega_c(V)}{\omega_{nc}(V)} , \tag{32}$$

where $\omega_c(V)$ denotes the first bending natural frequency of the cracked beam as a function of the volume fraction of fiber, $\omega_{nc}(\alpha)$ denotes the first bending natural frequency of the noncracked beam as a function of the volume fraction of fiber.

The decrease of the first bending natural frequency is a function of the volume fraction of fiber. The maximum value is achieved at relatively higher fiber fractions (around 40%).

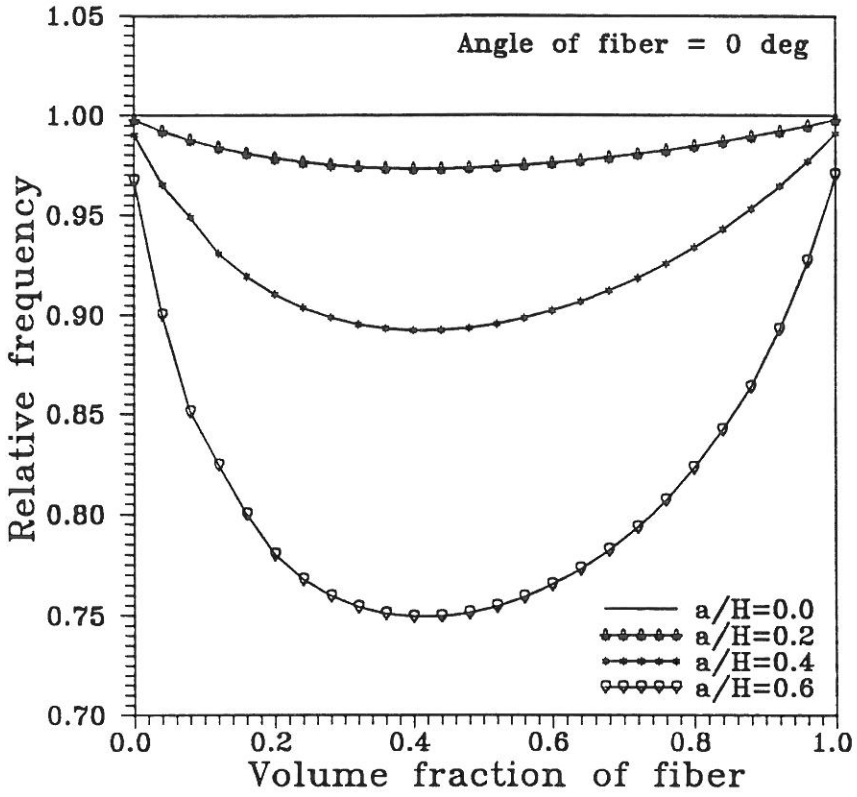


Fig.12

## 9. CONCLUSIONS

The paper presents a new beam finite element with the transverse nonpropagating one-edge open crack situated in the middle of its length. The element is versatile and can be used for static and dynamic analysis of composite or isotropic beam-like structures. In all cases, the results obtained with use of the element are satisfactory. The compatibility of the element with most plate and shell elements as a stiffener is apparent due its simple nodal variables of $C^0$-type. The method of creating the element, makes it possible to construct beam finite elements with various type of cracks (double-edge, internal etc.) if the stress intensity factors for a given type of crack are known.

The crack in the cantilever composite beam causes, as it was easily be expected, a increase of the static deflection and a reduce of the first bending natural frequency of its. These changes are a function not only the depth of the crack (like in the case of isotropic materials), but also the volume fraction of fiber and the angle of fiber. The

intensity of changes increases in accord with the increase of the depth of the crack. The changes of the static deflection and the fundamental natural frequency are largest when the volume fraction of fiber is equal to 40% and the crack is perpendicular to the fibers of the composite. For the angle of fiber greater than 45 deg. the static deflection and the first bending natural frequency have the similar value like in the case of the noncracked beam, (for all values of the volume fraction of fiber).

## 10. ACKNOWLEDGEMENTS

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## APPENDIX A

The properties of the graphite-fiber reinforced polyimide composite analyzed in the paper are assumed as follows [18]

Modulus of Elasticity: $E_m$ = 2.756 GPa, $E_f$ = 275.6 GPa,

Poisson Ratio: $\nu_m$ = 0.33, $\nu_f$ = 0.2,

Modulus of Rigidity: $G_m$ = 1.036 GPa, $G_f$ = 114.8 GPa,

Mass Density: $\rho_m$ = 1600 kg/m$^3$, $\rho_f$ = 1900 kg/m$^3$,

where m denotes the matrix whereas f denotes the fiber.

The material is assumed orthotropic with respect to its axes of symmetry which lie along and perpendicular to the direction of fibers. The gross mechanical properties of the composite are calculated using the following formulas [18]

$$\rho = \rho_f v + \rho_m(1-v) ,$$

$$E_{11} = E_f v + E_m(1-v) \ ,$$

$$E_{22} = E_m \left[\frac{E_f + E_m + (E_f - E_m)v}{E_f + E_m - (E_f - E_m)v}\right] \ ,$$

$$\nu_{12} = \nu_f v + \nu_m(1-v) \ ,$$

$$\nu_{23} = \nu_f v + \nu_m(1-v)\left[\frac{1 + \nu_m - \nu_{12}E_m/E_{11}}{1 - \nu_m^2 + \nu_m\nu_{12}E_m/E_{11}}\right] \ ,$$

$$G_{12} = G_m \left[\frac{G_f + G_m + (G_f - G_m)v}{G_f + G_m - (G_f - G_m)v}\right] \ ,$$

$$G_{23} = \frac{E_{22}}{2(1 + \nu_{23})} \ ,$$

where v denotes the volume fraction of fiber.

The principal axes 1 and 2 are in the plane of the composite specimen aligned along and perpendicular to the fibers directions.

## APPENDIX B

The complex constants $s_1$ and $s_2$ in relations (20.a-c) are roots of the following characteristic equation [19]

$$\bar{b}_{11}s^4 - 2\bar{b}_{16}s^3 + (2\bar{b}_{12} + \bar{b}_{66})s^2 - 2\bar{b}_{26}s + \bar{b}_{22} = 0 \ .$$

The constant $\bar{b}_{ij}$ are calculated from the following relations [19]

$$\bar{b}_{11} = b_{11}m^4 + (2b_{12} + b_{66})m^2n^2 + b_{22}n^4 \ ,$$

$$\bar{b}_{22} = b_{11}n^4 + (2b_{12} + b_{66})m^2n^2 + b_{22}m^4 \ ,$$

$$\bar{b}_{12} = (b_{11} + b_{22} - b_{66})m^2n^2 + b_{12}(m^4 + n^4) ,$$

$$\bar{b}_{16} = (-2b_{11} + 2b_{12} + b_{66})m^3n + (2b_{22} - 2b_{12} - b_{66})mn^3 ,$$

$$\bar{b}_{26} = (-2b_{11} + 2b_{12} + b_{66})n^3m + (2b_{22} - 2b_{12} - b_{66})nm^3 ,$$

$$\bar{b}_{66} = 2(2b_{11} - 4b_{12} + 2b_{22} - b_{66})m^2n^2 + b_{66}(m^4 + n^4) ,$$

where $m = \cos\alpha$, $n = \sin\alpha$ ($\alpha$ denotes the angle between geometric axes of the beam and the material principal axes) - see Fig.1.

The terms $b_{ij}$ corresponds with the situation when geometric axes of the beam coincide with material principal axes. These are related to the mechanical constants of the material by [19]

$$b_{11} = \frac{1}{E_{11}}\left(1 - \nu_{12}^2 \frac{E_{22}}{E_{11}}\right) ,$$

$$b_{22} = \frac{1}{E_{22}}\left(1 - \nu_{23}^2\right) ,$$

$$b_{12} = \frac{-\nu_{12}}{E_{11}}\left(1 + \nu_{23}\right) ,$$

$$b_{66} = \frac{1}{G_{12}} .$$

The roots of the characteristic equation are either complex or pure imaginary and cannot be real. Thus, the four roots separate into two sets of distinct complex conjugates. The parameters $s_1$ and $s_2$ correspond to those with positive imaginary parts. The roots of characteristic equation were computed with an accuracy of $10^{-10}$ using Newton-Raphson method for polynomial equations with complex roots.

## APPENDIX C

In the case of the analyzed element, the stress-strain relation matrix posses the form [18]

$$\mathbb{D} = \left[\begin{array}{c|c} \bar{S}_{11} & \bar{S}_{16} \\ \hline \bar{S}_{16} & \bar{S}_{66} \end{array}\right]$$

where [18]

$$\bar{S}_{11} = S_{11}m^4 + 2(S_{12} + 2S_{66})m^2n^2 + S_{22}n^4 ,$$

$$\bar{S}_{16} = (S_{11} - S_{12} - 2S_{66})m^3n + (S_{12} - S_{22} + 2S_{66})mn^3 ,$$

$$\bar{S}_{66} = (S_{11} - 2S_{12} + S_{22} - 2S_{66})m^2n^2 + S_{66}(m^4 + n^4) .$$

The terms $S_{ij}$ corresponding with the material principal axes are determined from the following relations [18]

$$S_{11} = \frac{E_{11}}{1 - \nu_{12}^2 \frac{E_{22}}{E_{11}}} ,$$

$$S_{22} = S_{11} \frac{E_{22}}{E_{11}} ,$$

$$S_{12} = \nu_{12} S_{22} ,$$

$$S_{66} = G_{12} .$$

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